

# Computing Spectral Elimination Ideals

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## Abstract

We present here an overview of the hypermatrix spectral decomposition deduced from the Bhattacharya-Mesner hypermatrix algebra [BM1, BM2]. We describe necessary and sufficient conditions for the existence of a spectral decomposition. We further extend to hypermatrices the notion of resolution of identity and use them to derive hypermatrix analog of matrix spectral bounds. Finally we describe an algorithm for computing generators of the spectral elimination ideals which considerably improves on Groebner basis computation suggested in [GER].

## 1 Introduction

This brief note discusses *hypermatrices*, a generalization of matrices which corresponds to a finite set of numbers each of which is indexed by a member of an integer cartesian product set of the form  $\{0, \dots, (n_0 - 1)\} \times \dots \times \{0, \dots, (n_{l-1} - 1)\}$ . Such a hypermatrix is said to be of order  $l$  and more conveniently called an  $l$ -hypermatrix. The algebra and the spectral analysis of hypermatrices arise from generalizations of familiar concepts of linear algebra. The reader is referred to [L] for a survey of important hypermatrix results. The reader is also referred to [LQ] for a detailed survey of the various approaches to the spectral analysis of hypermatrices. The algebra discussed here differs considerably from the hypermatrix algebras surveyed in [L]. The hypermatrix algebra discussed here centers around the Bhattacharya-Mesner hypermatrix product operation motivated by generalizations of association schemes introduced in [BM1, BM2, B] and followed up in [GER]. Although the scope of the Bhattacharya-Mesner algebra extends to hypermatrices of all integral orders, the present discussion will be restricted for notational convenience to 3-hypermatrices since all the result presented here generalize straight-forwardly to greater order hypermatrices.

Our main result provides necessary and sufficient conditions for the existence of a spectral decomposition for a given hypermatrix which is symmetric under cyclic permutation of it's indices. We also describe how to extend to hypermatrices the notion of resolution of identity deduced from orthogonal hypermatrices introduced in [GER]. We also extend to hypermatrices the symmetrization approach to Singular Value Decomposition. We further derive hypermatrix analog of matrix spectral bounds. Finally, we describe an algorithm for computing generators of the elimination ideals which considerably improves on Groebner basis computation approach suggested in [GER] for computing elimination ideals.

## 2 Hypermatrix orthogonality

We describe here a spectral decomposition for 3-hypermatrices, deduced from the Bhattacharya-Mesner algebra. The proposed spectral decomposition builds on the notion of hypermatrix orthogonality introduced in [GER], defined for arbitrary order hypermatrices. In particular, we establish the existence of arbitrary order orthogonal hypermatrices by describing an explicit parametrization orthogonal hypermatrices resulting from direct sums of hypermatrices of size  $2 \times 2 \times \dots \times 2 \times 2$  ( the direct sum refers here to hypermatrix diagonal block construction quite analogous to the matrix counterpart). We recall that hypermatrix orthogonality for an  $m$ -hypermatrix is determined by the constraints

$$\Delta = \bigcirc_{0 \leq t < m} \left( \mathbf{Q}^{T^{(m-t)}} \right) \quad (1)$$

which is more explicitly expressed as

$$\delta_{i_0 \dots i_{m-1}} = \sum_{0 \leq k < n} q_{i_0 k i_2 \dots i_{m-2} i_{m-1}} \cdot q_{i_1 k i_3 \dots i_{m-1} i_0} \cdot q_{i_2 k i_4 \dots i_0 i_1} \cdot \dots \cdot q_{i_{m-2} k i_0 \dots i_{m-4} i_{m-3}} \cdot q_{i_{m-1} k i_1 \dots i_{m-3} i_{m-2}}, \quad (2)$$

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(where  $\delta_{i_0 \dots i_{m-1}}$  denotes the entries of the Kronecker delta). It follows that the parametrization of the sought after family of direct sums of orthogonal hypermatrices is completely determined by the parametrization of hypermatrices of dimensions  $\underbrace{2 \times 2 \times \dots \times 2 \times 2}_m$  which we determine by solving the linear constraints of the form

$$\begin{aligned} & \ln q_{i_0 0 i_2 \dots i_{m-2} i_{m-1}} + \ln q_{i_1 0 i_3 \dots i_{m-1} i_0} + \ln q_{i_2 0 i_4 \dots i_0 i_1} + \dots + \ln q_{i_{m-2} 0 i_0 \dots i_{m-4} i_{m-3}} + \ln q_{i_{m-1} 0 i_1 \dots i_{m-3} i_{m-2}} = \\ & i\pi + \ln q_{i_0 1 i_2 \dots i_{m-2} i_{m-1}} + \ln q_{i_1 1 i_3 \dots i_{m-1} i_0} + \ln q_{i_2 1 i_4 \dots i_0 i_1} + \dots + \ln q_{i_{m-2} 1 i_0 \dots i_{m-4} i_{m-3}} + \ln q_{i_{m-1} 1 i_1 \dots i_{m-3} i_{m-2}}. \end{aligned} \quad (3)$$

Note that there will be one constraint for every orbit of the action of the cyclic group on  $m$ -tuples. For instance the constraints above yield the following parametrization for orthogonal hypermatrices of size  $2 \times 2 \times 2$  expressed by

$$q_{000} = \frac{e^{r_3}}{(e^{(3r_3)} + e^{(3r_6)})^{\frac{1}{3}}}, \quad q_{001} = e^{r_4}, \quad q_{010} = \frac{e^{r_6}}{(e^{(3r_3)} + e^{(3r_6)})^{\frac{1}{3}}}, \quad q_{011} = e^{r_2} \quad (4)$$

$$q_{100} = -e^{(r_2 - r_3 - r_4 + r_5 + r_6)}, \quad q_{101} = \frac{e^{(r_1 + r_3 - r_6)}}{(e^{(3r_1)} + e^{(3r_1 + 3r_3 - 3r_6)})^{\frac{1}{3}}}, \quad q_{110} = e^{r_5}, \quad q_{111} = \frac{e^{r_1}}{(e^{(3r_1)} + e^{(3r_1 + 3r_3 - 3r_6)})^{\frac{1}{3}}} \quad (5)$$

for arbitrary choice of values of parameters  $\{r_k\}_{0 < k < 7}$ .

### 3 Matrix spectral elimination ideals

We recall that the spectral constraint for symmetric real matrices are expressed as follows

$$\begin{cases} \mathbf{A} &= (\mathbf{Q} \cdot \mathbf{D}) \cdot (\mathbf{Q} \cdot \mathbf{D})^T \\ [\mathbf{Q} \cdot \mathbf{Q}^T]_{i,j} &= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \quad \forall 0 \leq i, j < n \\ \mathbf{D}^{\star^2} &= \mathbf{D}^T \cdot \mathbf{D} \end{cases} \quad (6)$$

and the corresponding invariance formulation is given by

$$\mathbf{A} \cdot [(\mathbf{Q} \cdot \mathbf{D})^T]^{-1} = \mathbf{Q} \cdot \mathbf{D}, \quad (7)$$

provided of course that the matrix  $(\mathbf{Q} \cdot \mathbf{D})^T$  is invertible. We describe here the determination of generators for the elimination ideals

$$I_{\mathbf{D}} = \langle \mathbf{A} - (\mathbf{Q} \cdot \mathbf{D}) \cdot (\mathbf{Q} \cdot \mathbf{D})^T, \mathbf{Q} \cdot \mathbf{Q}^T - \mathbf{\Delta}, \mathbf{D}^{\star^2} - \mathbf{D}^T \cdot \mathbf{D} \rangle \cap \mathbb{C}[\mathbf{D}] \quad (8)$$

and

$$I_{\mathbf{Q}} = \langle \mathbf{A} - (\mathbf{Q} \cdot \mathbf{D}) \cdot (\mathbf{Q} \cdot \mathbf{D})^T, \mathbf{Q} \cdot \mathbf{Q}^T - \mathbf{\Delta}, \mathbf{D}^{\star^2} - \mathbf{D}^T \cdot \mathbf{D} \rangle \cap \mathbb{C}[\mathbf{Q}] \quad (9)$$

where  $\langle \mathbf{A} - (\mathbf{Q} \cdot \mathbf{D}) \cdot (\mathbf{Q} \cdot \mathbf{D})^T, \mathbf{Q} \cdot \mathbf{Q}^T - \mathbf{\Delta}, \mathbf{D}^{\star^2} - \mathbf{D}^T \cdot \mathbf{D} \rangle$  denotes the ideal generated by the matrix spectral constraints. While the characterization of both these elimination ideals is well known in the case of matrices, our aim is to present the derivation of these elimination ideal so to suggest a natural generalization of the derivation to hypermatrices of all integral orders. We emphasize that the proposed derivation avoids the Groebner basis computations suggested in [GER].

As a starting point for the derivation we consider the following equivalent formulation of the matrix spectral constraints

$$\begin{cases} a_{ij} &= \langle (\mathbf{q}_i \star \boldsymbol{\lambda}), (\boldsymbol{\lambda} \star \mathbf{q}_j) \rangle \\ \delta_{ij} &= \langle \mathbf{q}_i, \mathbf{q}_j \rangle \end{cases} \quad \forall 0 \leq i \leq j < n \quad (10)$$

where  $\boldsymbol{\lambda}$  denotes the vector whose entries are the principal square roots of the eigenvalues of  $\mathbf{A}$ . We therefore deduce the following expressions for the columns of  $\mathbf{A}$

$$\mathbf{a}_j = (a_{ij} = \langle (\mathbf{q}_i \star \boldsymbol{\lambda}), (\boldsymbol{\lambda} \star \mathbf{q}_j) \rangle)_{0 \leq i < n} \Rightarrow \mathbf{a}_j = \mathbf{Q}^T \cdot (\boldsymbol{\lambda}^{\star^2} \star \mathbf{q}_j) \quad (11)$$

and in particular

$$\langle \mathbf{a}_i, \mathbf{a}_j \rangle = [\mathbf{Q}^T \cdot (\boldsymbol{\lambda}^{\star^2} \star \mathbf{q}_j)]^T \cdot [\mathbf{Q}^T \cdot (\boldsymbol{\lambda}^{\star^2} \star \mathbf{q}_i)] \quad (12)$$

$$\Rightarrow \langle \mathbf{a}_i, \mathbf{a}_j \rangle = \left\langle (\boldsymbol{\lambda}^{\star^2} \star \mathbf{q}_j), (\boldsymbol{\lambda}^{\star^2} \star \mathbf{q}_i) \right\rangle \quad (13)$$

quite similarly let  $\mathbf{a}_j^{[k-1]}$  denote the  $j$ -th column of the matrix power  $\mathbf{A}^{k-1}$ , we have

$$\left\langle \mathbf{a}_i, \mathbf{a}_j^{[k-1]} \right\rangle = \left\langle (\boldsymbol{\lambda}^{\star^k} \star \mathbf{q}_j), (\boldsymbol{\lambda}^{\star^k} \star \mathbf{q}_i) \right\rangle = \left\langle \boldsymbol{\lambda}^{\star^{2k}}, (\mathbf{q}_j \star \mathbf{q}_i) \right\rangle. \quad (14)$$

We therefore deduce Vandermonde block linear constraints which we conveniently express in matrix form as follows

$$\begin{pmatrix} [\mathbf{q}_i \star \mathbf{q}_j]_0 \\ \vdots \\ [\mathbf{q}_i \star \mathbf{q}_j]_{n-1} \end{pmatrix}_{0 \leq i \leq j < n} = \left( \mathbf{I}_{\binom{n+1}{2}} \otimes \mathbf{V}(\boldsymbol{\lambda}^{\star^2}) \right)^{-1} \cdot \begin{pmatrix} [\mathbf{A}^0]_{i,j} \\ \vdots \\ [\mathbf{A}^{n-1}]_{i,j} \end{pmatrix}_{0 \leq i \leq j < n} \quad (15)$$

where  $\mathbf{V}(\mathbf{x})$  denotes the  $n \times n$  Vandermonde matrix expressed by

$$\mathbf{V}(\mathbf{x}) := \left( v_{ij}(\mathbf{x}) = (x_j)^i \right)_{0 \leq i, j < n}. \quad (16)$$

Finally, the elimination ideal  $I_{\mathbf{D}}$  is determined by equating corresponding expressions to obtain  $\binom{n}{2}$  vector constraints determined by the equalities

$$\left\{ (\mathbf{q}_i \star \mathbf{q}_j)^{\star^2} = \mathbf{q}_i^{\star^2} \star \mathbf{q}_j^{\star^2} \right\}_{0 \leq i < j < n} \quad (17)$$

We have therefore derived generators for the ideal of elementary symmetric polynomials in the square roots of the eigenvalues of  $\mathbf{A}$ .

Although the derivation steps described in the previous paragraph are insightful for the matrix case, unfortunately, these derivation steps are of limited interest for higher order hypermatrices. In fact the derivation steps described here only extend to hypermatrices which correspond to direct sums of hypermatrices whose size is of the form  $2 \times 2 \times 2 \times \dots \times 2$ . Fortunately, however, the derivation of the elimination ideal  $I_{\mathbf{Q}}$  naturally extend to general hypermatrices. As a result, we advocate the use of the elimination ideal  $I_{\mathbf{Q}}$ , as a basis for iterative procedures for approximating the spectral decomposition of hypermatrices. Our starting point for the matrix case will be  $2\binom{n+1}{2}$  quadratic constraints

$$\begin{cases} a_{ij} &= \langle (\mathbf{q}_i \star \boldsymbol{\lambda}), (\boldsymbol{\lambda} \star \mathbf{q}_j) \rangle \\ \delta_{ij} &= \langle \mathbf{q}_i, \mathbf{q}_j \rangle \end{cases} \quad \forall 0 \leq i \leq j < n. \quad (18)$$

The main step of the derivation consists in combining the decomposition constraints with the orthogonality constraints via the use of induced resolutions of identity. We recall for the convenience of the reader that the resolution of identity induced by the orthogonal matrix  $\mathbf{Q}$  can be expressed as follows

$$\forall \mathbf{u}, \mathbf{v} \in \mathbb{C}^n, \quad \langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle_{(\sum_{0 \leq t < n} \mathbf{q}_t \cdot \mathbf{q}_t^T)} = \sum_{0 \leq t < n} \langle \mathbf{u}, \mathbf{v} \rangle_{(\mathbf{q}_t \cdot \mathbf{q}_t^T)}. \quad (19)$$

The resolution of identity property is precisely the reason why orthogonality plays such a crucial role in the formulation of the spectral constraints. Using the resolution of identity, we reduce the spectral constraints to the  $\binom{n+1}{2}$  constraints

$$\left\{ a_{ij} = \sum_{0 \leq k < n} \langle (\mathbf{q}_i \star \boldsymbol{\lambda}), (\boldsymbol{\lambda} \star \mathbf{q}_j) \rangle_{\mathbf{q}_k \cdot \mathbf{q}_k^T} \right\}_{0 \leq i \leq j < n} \quad (20)$$

which may more conveniently be rewritten as

$$\left\{ a_{ij} = \sum_{0 \leq k < n} \left\langle (\boldsymbol{\lambda} \cdot \boldsymbol{\lambda}^T), (\mathbf{q}_k \cdot \mathbf{q}_k^T) \star (\mathbf{q}_i \cdot \mathbf{q}_j^T) \right\rangle \right\}_{0 \leq i \leq j < n}. \quad (21)$$

We may think off the set of constraints above as  $\binom{n+1}{2}$  linear constraints in the  $\binom{n+1}{2}$  variables  $\{\lambda_i \lambda_j\}_{0 \leq i < j < n}$ . The system is thus solved via Cramer's rule and thus the elimination ideal  $I_{\mathbf{Q}}$  is obtain from equating the appropriate constraints, suggested by the equality

$$\left\{ (\lambda_i \lambda_j)^2 = \lambda_i^2 \lambda_j^2 \right\}_{0 \leq i < j < n}. \quad (22)$$

Clearly the ideal  $I_{\mathbf{Q}}$  is equivalently characterized by

$$\forall 0 \leq i < j < n, \quad [\mathbf{Q} \cdot \mathbf{A} \cdot \mathbf{Q}^T]_{i,j} = 0 \quad (23)$$

subject to the constraints

$$\mathbf{Q}^T \cdot \mathbf{Q} = \mathbf{I} \quad (24)$$

However the advantage of the proposed derivation of the elimination ideal  $I_{\mathbf{Q}}$  is that it completely determines the expression of the eigenvalues in terms of the eigenvectors and the generators for the constraint are half the size of the original constraints. We may further remark that the derivation suggest a natural mapping between the orthogonality constraints

$$\{\langle \mathbf{q}_i, \mathbf{q}_j \rangle = 0\}_{0 \leq i < j < n}$$

and the constraints which determines  $I_{\mathbf{Q}}$ , namely

$$\left\{ (\lambda_i \lambda_j)^2 - \lambda_i^2 \lambda_j^2 = 0 \right\}_{0 \leq i < j < n} \quad (25)$$

It is not unlikely that such a mapping may in off itself suggest alternative proof of existence and unicity of the spectral decomposition for symmetric matrices.

## 4 3-Hypermatrix elimination ideals.

We describe here how we extend to hypermatrices the elimination ideal computations describe in the previous section. We recall here that third order hypermatrix spectral constraints introduced in [GER] are expressed as

$$\left\{ \begin{array}{lcl} \mathbf{A} & = & \circ \left( \circ (\mathbf{Q}, \mathbf{D}, \mathbf{D}^T), \circ (\mathbf{Q}, \mathbf{D}, \mathbf{D}^T)^{T^2}, \circ (\mathbf{Q}, \mathbf{D}, \mathbf{D}^T)^T \right) \\ \left[ \circ (\mathbf{Q}, \mathbf{Q}^{T^2}, \mathbf{Q}^T) \right]_{i,j,k} & = & \begin{cases} 1 & \text{if } i = j = k \\ 0 & \text{otherwise} \end{cases} \quad \forall 0 \leq i, j, k < n \\ \mathbf{D}^{\star^3} & = & \circ (\mathbf{D}^T, \mathbf{D}^{T^2}, \mathbf{D}) \end{array} \right. \quad (26)$$

Just as we did for matrices, we may express for hypermatrices the corresponding invariance equality expressed by

$$\circ \left( \mathbf{A}, \left[ \circ (\mathbf{Q}, \mathbf{D}, \mathbf{D}^T)^{T^2} \right]^{(-1)_1}, \left[ \circ (\mathbf{Q}, \mathbf{D}, \mathbf{D}^T)^T \right]^{(-1)_2} \right) = \circ (\mathbf{Q}, \mathbf{D}, \mathbf{D}^T) \quad (27)$$

provided of course that the pair of hypermatrices  $\left( \circ (\mathbf{Q}, \mathbf{D}, \mathbf{D}^T)^{T^2}, \circ (\mathbf{Q}, \mathbf{D}, \mathbf{D}^T)^T \right)$  forms an invertible pair in the sense defined in [BM2]. As pointed out in the previous section, the approach for deriving the elimination ideal

$$I_{\mathbf{D}} = \left\langle \mathbf{A} - \circ \left( \circ (\mathbf{Q}, \mathbf{D}, \mathbf{D}^T), \circ (\mathbf{Q}, \mathbf{D}, \mathbf{D}^T)^{T^2}, \circ (\mathbf{Q}, \mathbf{D}, \mathbf{D}^T)^T \right), \mathbf{A} - \circ (\mathbf{Q}, \mathbf{Q}^{T^2}, \mathbf{Q}^T), \mathbf{D}^{\star^3} - \circ (\mathbf{D}^T, \mathbf{D}^{T^2}, \mathbf{D}) \right\rangle \cap \mathbb{C}[\mathbf{D}] \quad (28)$$

only extends to hypermatrices which are direct sums of hypermatrices whos size is of the form  $2 \times 2 \times 2 \times \dots \times 2$ . In the particular case of 3-hypermatrices the problem completely reduces to the spectral decomposition of  $2 \times 2 \times 2$  hypermatrices

determined by the constraints

$$\left\{ \begin{array}{ll} \left\langle \left( \mathbf{w}_0^{*6} \right), (\mathbf{q}_{00} \star \mathbf{q}_{00} \star \mathbf{q}_{00}) \right\rangle & = a_{000} \\ \left\langle \left( \mathbf{w}_1^{*6} \right), (\mathbf{q}_{11} \star \mathbf{q}_{11} \star \mathbf{q}_{11}) \right\rangle & = a_{111} \\ \left\langle \left( \mathbf{w}_0^{*2} \star \mathbf{w}_1^{*4} \right), (\mathbf{q}_{01} \star \mathbf{q}_{10} \star \mathbf{q}_{11}) \right\rangle & = a_{011} \\ \left\langle \left( \mathbf{w}_0^{*4} \star \mathbf{w}_1^{*2} \right), (\mathbf{q}_{10} \star \mathbf{q}_{01} \star \mathbf{q}_{00}) \right\rangle & = a_{100} \\ \left\langle \left( \mathbf{w}_0^{*6} \right)^{\star 0}, (\mathbf{q}_{00} \star \mathbf{q}_{00} \star \mathbf{q}_{00}) \right\rangle & = \delta_{000} \\ \left\langle \left( \mathbf{w}_1^{*6} \right)^{\star 0}, (\mathbf{q}_{11} \star \mathbf{q}_{11} \star \mathbf{q}_{11}) \right\rangle & = \delta_{111} \\ \left\langle \left( \mathbf{w}_0^{*2} \star \mathbf{w}_1^{*4} \right)^{\star 0}, (\mathbf{q}_{01} \star \mathbf{q}_{10} \star \mathbf{q}_{11}) \right\rangle & = \delta_{011} \\ \left\langle \left( \mathbf{w}_0^{*4} \star \mathbf{w}_1^{*2} \right)^{\star 0}, (\mathbf{q}_{10} \star \mathbf{q}_{01} \star \mathbf{q}_{00}) \right\rangle & = \delta_{100} \end{array} \right. . \quad (29)$$

The system of equations above corresponds to a block Vandermonde set of linear constraints which yields the following constraints

$$\left\{ \begin{array}{ll} \frac{(w_{00}^4 w_{01}^2 - w_{01}^4 w_{11}^2)^3 (a_{000} - w_{01}^6)}{a_{001}^3 (w_{00}^6 - w_{01}^6)} & = \frac{(w_{00}^2 w_{01}^4 - w_{01}^2 w_{11}^4)^3 (a_{111} - w_{11}^6)}{a_{011}^3 (w_{01}^6 - w_{11}^6)} \\ \frac{(w_{00}^4 w_{01}^2 - w_{01}^4 w_{11}^2)^3 (w_{00}^6 - a_{000})}{a_{001}^3 (w_{00}^6 - w_{01}^6)} & = \frac{(w_{00}^2 w_{01}^4 - w_{01}^2 w_{11}^4)^3 (w_{01}^6 - a_{111})}{a_{011}^3 (w_{01}^6 - w_{11}^6)} \end{array} \right. \quad (30)$$

from which we deduce that the characteristic polynomial for  $2 \times 2 \times 2$  hypermatrices is given by

$$(w_{00}^6 w_{11}^6 - w_{01}^{12}) + w_{01}^6 (a_{000} + a_{111}) - (a_{111} w_{00}^6 + a_{000} w_{11}^6) . \quad (31)$$

Incidentally it immediately follows that characteristic polynomials of direct sums of  $2 \times 2 \times 2$  matrices is determined by the derivation described above.

In order to derive the elimination ideal  $I_{\mathbf{Q}}$  using the hypermatrix formulation of the resolution identity we will consider the sequence of hypermatrices defined as follows

$$\mathbf{U}_0 = \mathbf{\Delta}, \quad \mathbf{U}_{k+1} = \circ_{\mathbf{U}_k} (\mathbf{Q}, \mathbf{Q}^{T^2}, \mathbf{Q}^T) \quad (32)$$

where  $\mathbf{\Delta}$  denotes the Kronecker delta and the ternary product determining  $\mathbf{U}_{k+1}$  corresponds to the hypermatrix product with background hypermatrix  $\mathbf{U}_k$  as introduced in [GER]. The  $k$ -th term of the recurrence yields  $n + 2\binom{n}{2} + 2\binom{n}{3}$  constraints, furthermore we know that for the purposes of elimination, the number of variables being considered equals  $n(n + 2\binom{n}{2} + 2\binom{n}{3})$ , it therefore follows that it is enough to compute a sequence of length  $n$ . Using the constraints

$$\left\{ \mathbf{A} = \circ_{\mathbf{U}_k} \left( \circ(\mathbf{Q}, \mathbf{D}, \mathbf{D}^T), \circ(\mathbf{Q}, \mathbf{D}, \mathbf{D}^T)^{T^2}, \circ(\mathbf{Q}, \mathbf{D}, \mathbf{D}^T)^T \right) \right\}_{0 \leq k < n} \quad (33)$$

in conjunction with Cramer's rule we express the monomials in the entries of  $\mathbf{D}$  as rational function in the entries of  $\mathbf{Q}$ , just as we did for matrices. Finally the elimination ideal  $I_{\mathbf{Q}}$  is determined by the constraints of the form

$$\left\{ \begin{array}{ll} \left( \lambda_{ki}^4 \lambda_{kp}^2 \right)^3 & = \left[ (\lambda_{ki}^2)^3 \right]^2 \left[ (\lambda_{kp}^2)^3 \right] \\ \left( \lambda_{ki}^2 \lambda_{kj}^2 \lambda_{kp}^2 \right)^3 & = (\lambda_{ki}^2)^3 (\lambda_{kj}^2)^3 (\lambda_{kp}^2)^3 \end{array} \right. , \forall (i, j, p) \in \left\{ \underbrace{\binom{l}{2}}_{(i, j, j)} \cup \underbrace{\binom{l}{2}}_{(i, i, j)} \cup \underbrace{\binom{l}{3}}_{(i, j, k)} \cup \underbrace{\binom{l}{3}}_{(j, i, k)} \right\} \quad (34)$$

which determines the elimination ideal  $I_{\mathbf{Q}}$ . Incidentally the necessary and sufficient condition for the existence of a spectral decomposition for a given symmetric hypermatrix  $\mathbf{A}$  is the fact that the elimination ideal  $I_{\mathbf{Q}}$  is non-trivial.

#### 4.1 Spectral bound.

We recall that for a positive definite symmetric matrix  $\mathbf{A}$  whose spectral decomposition is expressed by

$$\begin{cases} \mathbf{A} &= (\mathbf{Q} \cdot \sqrt{\mathbf{D}}) \cdot (\mathbf{Q} \cdot \sqrt{\mathbf{D}})^T \\ [\mathbf{Q} \cdot \mathbf{Q}^T]_{i,j} &= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \quad \forall 0 \leq i, j < n \\ \mathbf{D}^{\star^2} &= \mathbf{D}^T \cdot \mathbf{D} \end{cases}, \quad (35)$$

we have that

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{C}^n \quad \langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{A}} := \left( \sum_{0 \leq i, j < n} a_{ij} x_i y_j \right) = \sum_{0 \leq k < n} \left\langle \sqrt{\lambda_k} \mathbf{x}, \sqrt{\lambda_k} \mathbf{y} \right\rangle_{\mathbf{q}_k \otimes \mathbf{q}_k} \quad (36)$$

it therefore follows from the resolution of identity that

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{C}^n \quad \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{0 \leq k < n} \langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{q}_k \otimes \mathbf{q}_k} \quad (37)$$

if the vectors  $\mathbf{x}$  and  $\mathbf{y}$  are chosen such that

$$\forall 0 \leq k < n, \quad \langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{q}_k \otimes \mathbf{q}_k} \geq 0$$

then the following spectral inequality holds

$$\left\langle \sqrt{\lambda_0} \mathbf{x}, \sqrt{\lambda_0} \mathbf{y} \right\rangle \leq \langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{A}} \leq \left\langle \sqrt{\lambda_n} \mathbf{x}, \sqrt{\lambda_n} \mathbf{y} \right\rangle \quad (38)$$

Similarly for some 3-hypermatrix  $\mathbf{A}$  with entries symmetric under cyclic permutation the corresponding spectral decomposition is expressed by

$$\begin{cases} \mathbf{A} &= \circ \left( \circ (\mathbf{Q}, \mathbf{D}, \mathbf{D}^T), \circ (\mathbf{Q}, \mathbf{D}, \mathbf{D}^T)^{T^2}, \circ (\mathbf{Q}, \mathbf{D}, \mathbf{D}^T)^T \right) \\ \left[ \circ (\mathbf{Q}, \mathbf{Q}^{T^2}, \mathbf{Q}^T) \right]_{i,j,k} &= \begin{cases} 1 & \text{if } i = j = k \\ 0 & \text{otherwise} \end{cases} \quad \forall 0 \leq i, j, k < n \\ \mathbf{D}^{\star^3} &= \circ (\mathbf{D}^T, \mathbf{D}^{T^2}, \mathbf{D}) \end{cases}. \quad (39)$$

We consider the very particular case where the scaling entries of  $\mathbf{D}$  are such that  $\forall 0 \leq j_0 < j_1 < n, \quad 0 \leq d_{j_0}(i) \leq d_{j_1}(i)$ , we have that

$$\begin{aligned} &\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{C}^n, \\ &\langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle_{\mathbf{A}} = \sum_{0 \leq k < n} \langle (\mathbf{d}_k \star \mathbf{x}), (\mathbf{d}_k \star \mathbf{y}), (\mathbf{d}_k \star \mathbf{z}) \rangle_{\otimes (Q_k, Q_k, Q_k)} \end{aligned} \quad (40)$$

$Q_k$  denote the  $k$ -th eigematrix of  $\mathbf{A}$ ,  $\otimes (Q_k, Q_k, Q_k)$  denote the matrix outer product as defined in [GER] and  $\mathbf{d}_k \star \mathbf{x}$  denotes the Hadamard product of the vectors  $\mathbf{d}_k$  and  $\mathbf{x}$ . Furthermore the hypermatrix resolution of identity associated with the orthogonal hypermatrix  $\mathbf{Q}$  is expressed by

$$\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{C}^n, \quad \langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle = \sum_{0 \leq j < n} \langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle_{\otimes (Q_j, Q_j, Q_j)} \quad (41)$$

in particular, if the vectors  $\mathbf{x}$ ,  $\mathbf{y}$  and  $\mathbf{z}$  are chosen such that

$$\forall 0 \leq k < n, \quad \langle (\mathbf{d}_k \star \mathbf{x}), (\mathbf{d}_k \star \mathbf{y}), (\mathbf{d}_k \star \mathbf{z}) \rangle_{\otimes (Q_k, Q_k, Q_k)} \geq 0$$

then the following spectral inequality holds

$$\langle (\mathbf{d}_0 \star \mathbf{x}), (\mathbf{d}_0 \star \mathbf{y}), (\mathbf{d}_0 \star \mathbf{z}) \rangle \leq \langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle_{\mathbf{A}} \leq \langle (\mathbf{d}_n \star \mathbf{x}), (\mathbf{d}_n \star \mathbf{y}), (\mathbf{d}_n \star \mathbf{z}) \rangle \quad (42)$$

which generalizes the matrix spectral inequality.

## 4.2 3-hypermatrix SVD

We now describe a natural generalization of symmetrization approach to hypermatrix SVD. We start with some arbitrary  $n \times n \times n$  hypermatrix  $\mathbf{A}$ , and deduce at most 3 symmetric 3-hypermatrices respectively given by  $\circ(\mathbf{A}, \mathbf{A}^{T^2}, \mathbf{A}^T)$ ,  $\circ(\mathbf{A}^T, \mathbf{A}, \mathbf{A}^{T^2})$  and  $\circ(\mathbf{A}^{T^2}, \mathbf{A}^T, \mathbf{A})$ . Furthermore as suggested by the spectral decomposition of 3-hypermatrices which are symmetric under cyclic permutations of their indices we are led to consider the following decomposition expressions associated with each symmetric hypermatrices.

$$\left\{ \begin{array}{lcl} \circ(\mathbf{A}, \mathbf{A}^{T^2}, \mathbf{A}^T) & = & \circ(\circ(\mathbf{Q}, \mathbf{D}, \mathbf{D}^T), \circ(\mathbf{Q}, \mathbf{D}, \mathbf{D}^T)^{T^2}, \circ(\mathbf{Q}, \mathbf{D}, \mathbf{D}^T)^T) \\ [\circ(\mathbf{Q}, \mathbf{Q}^{T^2}, \mathbf{Q}^T)]_{i,j,k} & = & \begin{cases} 1 & \text{if } i = j = k \\ 0 & \text{otherwise} \end{cases} \quad \forall 0 \leq i, j, k < n \\ \mathbf{D}^{\star^3} & = & \circ(\mathbf{D}^T, \mathbf{D}^{T^2}, \mathbf{D}) \end{array} \right. \quad (43)$$

$$\left\{ \begin{array}{lcl} \circ(\mathbf{A}^T, \mathbf{A}, \mathbf{A}^{T^2}) & = & \circ(\circ(\mathbf{E}, \mathbf{U}, \mathbf{E}^{T^2})^T, \circ(\mathbf{E}, \mathbf{U}, \mathbf{E}^{T^2}), \circ(\mathbf{E}, \mathbf{U}, \mathbf{E}^{T^2})^{T^2}) \\ [\circ(\mathbf{U}^T, \mathbf{U}, \mathbf{U}^{T^2})]_{i,j,k} & = & \begin{cases} 1 & \text{if } i = j = k \\ 0 & \text{otherwise} \end{cases} \quad \forall 0 \leq i, j, k < n \\ \mathbf{E}^{\star^3} & = & \circ(\mathbf{E}^T, \mathbf{E}^{T^2}, \mathbf{E}) \end{array} \right. \quad (44)$$

$$\left\{ \begin{array}{lcl} \circ(\mathbf{A}^{T^2}, \mathbf{A}^T, \mathbf{A}) & = & \circ(\circ(\mathbf{F}^T, \mathbf{F}^{T^2}, \mathbf{V})^{T^2}, \circ(\mathbf{F}^T, \mathbf{F}^{T^2}, \mathbf{V})^T, \circ(\mathbf{F}^T, \mathbf{F}^{T^2}, \mathbf{V})) \\ [\circ(\mathbf{V}^{T^2}, \mathbf{V}^T, \mathbf{V})]_{i,j,k} & = & \begin{cases} 1 & \text{if } i = j = k \\ 0 & \text{otherwise} \end{cases} \quad \forall 0 \leq i, j, k < n \\ \mathbf{F}^{\star^3} & = & \circ(\mathbf{F}^T, \mathbf{F}^{T^2}, \mathbf{F}) \end{array} \right. \quad (45)$$

Incidentally the framework for the symmetrization approach results from a desired product of the form

$$\circ(\circ(\mathbf{Q}, \mathbf{D}, \mathbf{D}^T), \circ(\mathbf{E}, \mathbf{U}, \mathbf{E}^{T^2}), \circ(\mathbf{F}^T, \mathbf{F}^{T^2}, \mathbf{V})). \quad (46)$$

Let

$$\begin{aligned} \tilde{\mathbf{Q}} &= \circ(\mathbf{Q}, \mathbf{D}, \mathbf{D}^T) \\ \tilde{\mathbf{E}} &= \circ(\mathbf{E}, \mathbf{U}, \mathbf{E}^{T^2}) \\ \tilde{\mathbf{F}} &= \circ(\mathbf{F}^T, \mathbf{F}^{T^2}, \mathbf{V}) \end{aligned}$$

Ideally we would want to have

$$\mathbf{A} = \circ(\tilde{\mathbf{Q}}, \tilde{\mathbf{E}}, \tilde{\mathbf{F}}) \quad (47)$$

but for most purposes we would be permit  $n$  additional parameters  $\{\alpha_t\}_{0 \leq t < n}$ , which are to be solve in the least square sense so as to yield an approximation of  $\mathbf{A}$

$$\mathbf{A} \approx \sum_{0 \leq k < n} \alpha_k \otimes (\tilde{Q}_k, \tilde{E}_k, \tilde{F}_k) \quad (48)$$

## 5 General Hypermatrix Spectral Decomposition

Just as matrices which are not symmetric admit a spectral decomposition, 3-hypermatrices which are not symmetric under cyclic permutation of their indices also admit a spectral decomposition. We shall discuss here the spectral decomposition of non-symmetric 3-hypermatrices. We show here how to extend to arbitrary hypermatrices, the derivation of the elimination ideals. For simplicity let us start with the matrix case. We recall that matrix spectral constraint for symmetric real matrices

are expressed by

$$\begin{cases} \mathbf{A} &= (\mathbf{U} \cdot \mathbf{D}) \cdot (\mathbf{V} \cdot \mathbf{D})^T \\ [\mathbf{U} \cdot \mathbf{V}^T]_{i,j} &= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \quad \forall 0 \leq i, j < n \\ \mathbf{D}^{\star^2} &= \mathbf{D}^T \cdot \mathbf{D} \end{cases} \quad (49)$$

We describe here the computation of generators for the elimination ideals

$$I_{\mathbf{U}, \mathbf{V}} = \langle \mathbf{A} - (\mathbf{U} \cdot \mathbf{D}) \cdot (\mathbf{V} \cdot \mathbf{D})^T, \mathbf{U} \cdot \mathbf{V}^T - \mathbf{\Delta}, \mathbf{D}^{\star^2} - \mathbf{D}^T \cdot \mathbf{D} \rangle \cap \mathbb{C}[\mathbf{U}, \mathbf{V}] \quad (50)$$

where  $\langle \mathbf{A} - (\mathbf{U} \cdot \mathbf{D}) \cdot (\mathbf{V} \cdot \mathbf{D})^T, \mathbf{U} \cdot \mathbf{V}^T - \mathbf{\Delta}, \mathbf{D}^{\star^2} - \mathbf{D}^T \cdot \mathbf{D} \rangle$  denotes the ideal generated by the matrix spectral constraints. Without any loss of generality we may write the spectral constraints as

$$\begin{cases} a_{ij} &= \langle (\mathbf{u}_i \star \boldsymbol{\lambda}), (\boldsymbol{\gamma} \star \mathbf{v}_j) \rangle \\ \delta_{ij} &= \langle \mathbf{u}_i, \mathbf{v}_j \rangle \end{cases} \quad \forall 0 \leq i, j < n \quad (51)$$

Using the resolution of identity, we reduce the spectral constraints to the following  $n^2$  constraints of the form

$$a_{ij} = \sum_{0 \leq k < n} \langle (\mathbf{u}_i \star \boldsymbol{\lambda}), (\boldsymbol{\gamma} \star \mathbf{v}_j) \rangle_{\mathbf{u}_k \cdot \mathbf{v}_k^T} \quad (52)$$

which may more conveniently be rewritten as

$$a_{ij} = \sum_{0 \leq k < n} \langle (\boldsymbol{\lambda} \cdot \boldsymbol{\gamma}^T), (\mathbf{u}_k \cdot \mathbf{v}_k^T) \star (\mathbf{u}_i \cdot \mathbf{v}_j^T) \rangle \quad (53)$$

which we think of as linear system of  $n^2$  constraints in linear in the  $n^2$  variables  $\{\lambda_i \gamma_j\}_{0 \leq i, j < n}$ .

Similarly for general hypermatrices the constraints is expressed by

$$\begin{cases} \mathbf{A} &= \circ \left( \circ (\mathbf{Q}, \mathbf{D}_0, \mathbf{D}_0^T), \circ (\mathbf{D}_1, \mathbf{U}, \mathbf{D}_1^{T^2}), \circ (\mathbf{D}_2^T, \mathbf{D}_2^{T^2}, \mathbf{V}) \right) \\ [\circ (\mathbf{Q}, \mathbf{U}, \mathbf{V})]_{i,j,k} &= \begin{cases} 1 & \text{if } i = j = k \\ 0 & \text{otherwise} \end{cases} \quad \forall 0 \leq i, j, k < n \\ \mathbf{D}_l^{\star^3} &= \circ (\mathbf{D}_l^T, \mathbf{D}_l^{T^2}, \mathbf{D}_l) \quad 0 \leq l < 3 \end{cases} \quad (54)$$

hence using the sequence

$$\mathbf{G}_0 = \mathbf{\Delta}, \quad \mathbf{G}_{k+1} = \circ_{\mathbf{G}_k} (\mathbf{Q}, \mathbf{U}, \mathbf{V}) \quad (55)$$

Using the constraints

$$\left\{ \mathbf{A} = \circ_{\mathbf{G}_k} \left( \circ (\mathbf{Q}, \mathbf{D}_0, \mathbf{D}_0^T), \circ (\mathbf{D}_1, \mathbf{U}, \mathbf{D}_1^{T^2}), \circ (\mathbf{D}_2^T, \mathbf{D}_2^{T^2}, \mathbf{V}) \right) \right\}_{0 \leq k < n} \quad (56)$$

in conjunction with Cramer's rule just as we did for symmetric hypermatrices we compute the elimination ideal for hypermatrices and deduce from it a criteria for the existence of a spectral decomposition.

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